JOURNAL OF COMPUTATIONAL PHYSICS 49, 265–275 (1983)

Strong Coupling Expansion of the Generating Functional for Gauge Systems on a Lattice with Arbitrary Sources

JAAP HOEK

Institute of Theoretical Physics, University of Amsterdam, 1018 XE Amsterdam, The Netherlands

Received March 17, 1982

A set of programs to calculate algebraically the generating functional (free energy) of a gauge system with arbitrary external sources on a lattice has been developed. It makes use of the strong coupling expansion. For theories with the standard $\mathrm{Tr}(UUU^{\dagger}U^{\dagger})$ action results have been obtained up to fourth order.

0. INTRODUCTION

The project described in this paper is part of a program aimed at making explicit calculations in quantum chromodynamics. As described in [1, 2] it uses the Euclidean version of the action on a four-dimensional lattice, where the generating functional for n-point functions turns into a free energy. The gauge fields are integrated out and the remaining integral is rewritten in terms of elementary meson and baryon fields. Thus we obtain an effective Lagrangian with arbitrary external sources. We can then proceed to calculate meson and baryon masses, couplings, and scattering amplitudes (e.g., $\rho\pi\pi$ coupling and $\pi\pi$ scattering). Hadron masses as well as scattering amplitudes have been obtained in [1, 2] in the strong coupling limit, $1/g^2 \to 0$. The continuum limit lies at weak coupling, $1/g^2 \to \infty$. A way to reach this region is the strong coupling expansion of the free energy together with extrapolation schemes such as Padé approximation supplemented by information from the renormalization group.

In this paper we develop a method to calculate the higher order terms of the expansion of the free energy

$$W(\beta, A, \bar{A}) = \log \int DU \exp\left\{\beta \sum_{\mathscr{P}} \mathrm{Tr}\, U_{\mathscr{P}} + \sum_{L} \mathrm{Tr}(\bar{A}_L U_L + A_L U_L^{\dagger})\right\} \tag{0.1}$$

in terms of the coupling parameter $\beta = 1/g^2$. In this expression L denotes a link, $\mathscr{P}$ denotes a plaquette and $\mathrm{Tr}\, U_{\mathscr{P}}$ is the standard plaquette action defined in Section 1. Furthermore, A and $\bar{A}$ are arbitrary external sources for the gauge fields. In a later stage they will be replaced by bilinear combinations of fermion fields coming from the gauge field–fermion interaction and the resulting expression will be rewritten in terms of meson and baryon fields.

0021-9991/83/020265–11$03.00/0

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The coefficients in the strong coupling expansion will be expressed in terms of derivatives of the one link free energy

$$W_L = \log \int DU_L \exp \operatorname{tr}(\bar{A}_L U_L + A_L U_L^\dagger). \tag{0.2}$$

These one-link integrals are known for a number of gauge groups [3, 4].

1. The Basic Formulae

We start with some notation. Greek letters ($\mu, \nu, \ldots$) will be used to denote lattice directions and a^μ, a^ν will be used for unit lattice vectors in these directions. Lattice points are denoted by $m, n, \ldots$, and links by L or explicitly $(n, n + a^\mu)$ or (n, μ). For plaquettes (cf. Fig. 1) the generic letter $\mathscr{P}$ or the explicit expression (n, μ, ν) will be used. Thus (n, μ, ν) denotes the oriented plaquette starting at lattice site n with directions μ and ν, i.e., $(n, n + a^\mu, n + a^\mu + a^\nu, n + a^\nu)$. Then the total gauge action is

$$\beta \sum_{n,\mu,\nu \neq \mu} \operatorname{Tr} U_{(n,\mu,\nu)} \equiv \beta \sum_{\mathscr{P}} \operatorname{Tr} U_{\mathscr{P}}, \tag{1.1}$$

where

$$U_{(n,\mu,\nu)} = U(n, \mu)\, U(n + a^\mu, \nu)\, U^\dagger(n + a^\nu, \mu)\, U^\dagger(n, \nu) \tag{1.2}$$

and $U(n, \mu)$ is the gauge matrix $(U^a b)$ sitting on link $(n, n + a^\mu)$ [5].

By abbreviating the measure $\prod_{\text{links } L} d\mu_{\text{Haar}}(U_L)$ to $D[U]$ we may write the generating functional with sources A and $\bar{A}$ residing on links as

$$Z(\beta, A, \bar{A}) = \int D[U] \exp \left\{ \beta \sum_{\mathscr{P}} \operatorname{Tr} U_{\mathscr{P}} + \sum_L \operatorname{Tr}(\bar{A}_L U_L + A_L U_L^\dagger) \right\}. \tag{1.3}$$

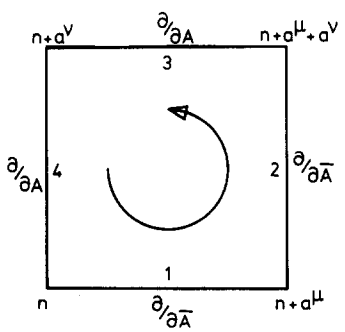

FIG. 1. A plaquette $\mathscr{P} = (n, \mu, \nu)$ with a numbering of its links. The differentiation symbols are explained in the text.

This generating functional satisfies the equation

$$\frac{\partial Z(\beta, A, \bar{A})}{\partial \beta} = \left[\sum_{\mathscr{P}} \mathrm{Tr}\left(\frac{\delta}{\delta A}\right)_{\mathscr{P}}\right] Z(\beta, A, \bar{A}), \tag{1.4}$$

where $\mathrm{Tr}(\delta/\delta A)_{\mathscr{P}}$ denotes the following differentiation:

$$\mathrm{Tr}\left(\frac{\delta}{\delta A}\right)_{\mathscr{P}=(n,\mu,\nu)} = \frac{\partial}{\partial A^a_b(n,\nu)} \frac{\partial}{\partial A^b_c(n+a^\nu,\mu)} \frac{\partial}{\partial \bar{A}^c_d(n+a^\mu,\nu)} \frac{\partial}{\partial \bar{A}^d_a(n,\mu)}. \tag{1.5}$$

We now substitute

$$Z(\beta, A, \bar{A}) = \exp W(\beta, A, \bar{A}) \tag{1.6}$$

in (1.4), and make a strong coupling expansion for W

$$W(\beta, A, \bar{A}) = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} W_k(A, \bar{A}). \tag{1.7}$$

Identification of equal powers of β in the left- and right-hand sides of (1.4) then leads to a recursion relation for the coefficients W_k. At this point we want to make a brief comment on the reason for using W directly. The recursion relation for W is highly nonlinear, in contrast with that for Z. Yet the way of making a strong coupling expansion for Z,

$$Z(\beta, A, \bar{A}) = \sum_{k=0}^{\infty} \beta^k Z_k(A, \bar{A}), \tag{1.8}$$

solving the linear recursion relation and finally taking the logarithm of Z to get W is no shortcut. The reason is that the coefficients Z_k will involve multiple sums over lattice points, i.e., they have all kinds of disconnected contributions, whereas the W_k will be seen in the next section to involve only one lattice sum, thus to have only connected contributions (this phenomenon is well known from standard weak coupling perturbation theory).

To be able to give the recursion relation for W_k in a concise form we make the abbreviations

$$\begin{aligned} &\mathscr{D}^b_a = \partial/\partial A^a_b(n,\nu), && \mathscr{D}^c_b = \partial/\partial A^b_c(n+a^\nu,\mu), \\ &\mathscr{D}^d_c = \partial/\partial \bar{A}^c_d(n+a^\mu,\nu), && \mathscr{D}^a_d = \partial/\partial \bar{A}^d_a(n,\mu). \end{aligned} \tag{1.9}$$

With these the recursion relation reads

$$
\begin{aligned}
W_{k+1} = \sum_{(n,\mu,\nu)} \Big\{ & \mathscr{D}_a^b \mathscr{D}_b^c \mathscr{D}_c^d \mathscr{D}_d^a W_k \\
& + \sum_{m_1=0}^{k} \binom{k}{m_1} [\mathscr{D}_a^b \mathscr{D}_b^c \mathscr{D}_c^d W_{m_1} \mathscr{D}_d^a W_{k-m_1} + \mathscr{D}_a^b \mathscr{D}_b^c \mathscr{D}_d^a W_{m_1} \mathscr{D}_c^d W_{k-m_1} \\
& + \mathscr{D}_a^b \mathscr{D}_c^d \mathscr{D}_d^a W_{m_1} \mathscr{D}_b^c W_{k-m_1} + \mathscr{D}_b^c \mathscr{D}_c^d \mathscr{D}_d^a W_{m_1} \mathscr{D}_a^b W_{k-m_1} \\
& + \mathscr{D}_a^b \mathscr{D}_b^c W_{m_1} \mathscr{D}_c^d \mathscr{D}_d^a W_{k-m_1} + \mathscr{D}_a^b \mathscr{D}_c^d W_{m_1} \mathscr{D}_b^c \mathscr{D}_d^a W_{k-m_1} \\
& + \mathscr{D}_a^b \mathscr{D}_d^a W_{m_1} \mathscr{D}_b^c \mathscr{D}_c^d W_{k-m_1}] \\
& + \sum_{m_1=0}^{k} \binom{k}{m_1} \sum_{m_2=0}^{k-m_1} \binom{k-m_1}{m_2} [\mathscr{D}_a^b W_{m_1} \mathscr{D}_b^c W_{m_2} \mathscr{D}_c^d \mathscr{D}_d^a W_{k-m_1-m_2} \\
& + \mathscr{D}_a^b W_{m_1} \mathscr{D}_c^d W_{m_2} \mathscr{D}_b^c \mathscr{D}_d^a W_{k-m_1-m_2} + \mathscr{D}_a^b W_{m_1} \mathscr{D}_d^a W_{m_2} \mathscr{D}_b^c \mathscr{D}_c^d W_{k-m_1-m_2} \\
& + \mathscr{D}_b^c W_{m_1} \mathscr{D}_c^d W_{m_2} \mathscr{D}_a^b \mathscr{D}_d^a W_{k-m_1-m_2} + \mathscr{D}_b^c W_{m_1} \mathscr{D}_d^a W_{m_2} \mathscr{D}_a^b \mathscr{D}_c^d W_{k-m_1-m_2} \\
& + \mathscr{D}_c^d W_{m_1} \mathscr{D}_d^a W_{m_2} \mathscr{D}_a^b \mathscr{D}_b^c W_{k-m_1-m_2}] \\
& + \sum_{m_1=0}^{k} \binom{k}{m_1} \sum_{m_2=0}^{k-m_1} \binom{k-m_1}{m_2} \sum_{m_3=0}^{k-m_1-m_2} \binom{k-m_1-m_2}{m_3} \\
& \times \mathscr{D}_a^b W_{m_1} \mathscr{D}_b^c W_{m_2} \mathscr{D}_c^d W_{m_3} \mathscr{D}_d^a W_{k-m_1-m_2-m_3} \Big\}. \qquad (1.10)
\end{aligned}
$$

The lowest order contribution $W_0(A, \bar{A})$ is readily identified. For $\beta = 0$ the contributions to $Z(\beta, A, \bar{A})$ in (1.3) on different links decouple and Z becomes a product of one-link integrals,

$$Z(0, A, \bar{A}) = \prod_L Z_0(A_L, \bar{A}_L), \qquad (1.11a)$$

$$Z_0(A_L, \bar{A}_L) = \int d\mu(U_L) \exp\{\mathrm{Tr}(\bar{A}_L U_L + A_L U_L^\dagger)\}. \qquad (1.11b)$$

Thus $W_0(A, \bar{A})$ is only a sum over one-link contributions

$$W_0(A, \bar{A}) = \sum_L F(A_L, \bar{A}_L), \qquad (1.12a)$$

$$F(A_L, \bar{A}_L) = \log Z_0(A_L, \bar{A}_L). \qquad (1.12b)$$

For $SU(3)$ an expansion of F in terms of group invariants was computed in [4]. The next contribution W_1 contains only one term and reads

$$W_1(A, \bar{A}) = \sum_{\substack{\text{plaquette} \\ (n,\mu,\nu)}} \frac{\partial F}{\partial A_b^a(n,\nu)} \frac{\partial F}{\partial A_c^b(n+a^\nu,\mu)} \frac{\partial F}{\partial \bar{A}_d^c(n+a^\mu,\nu)} \frac{\partial F}{\partial \bar{A}_a^d(n,\mu)}. \qquad (1.13)$$

In the following sections we show how to implement the recursion relation as a set of computer algorithms.

2. Implementation on a Computer

Recursion relation (1.10) expresses W_k as a sum over plaquettes of sums of products of functional derivatives of lower order contributions, which may seem to result in a very complex structure of the higher order terms. We shall show, however, that in each order there is only one generic sum over the lattice points and that the number of lattice directions to be summed over can be kept small.

Concerning the first point, we shall show that W_k has the form (in a very condensed notation)

$$W_k = \sum_n \sum_M c_M^k \sum_{\mu_1,\ldots,\mu_l} \prod_i G(n + b_i, v_i). \tag{2.1}$$

In (2.1) there is one summation over the lattice points n and there are summations over lattice directions $\mu_1,\ldots,\mu_l$; a finite number of different contributions is possible, labelled by M. The contributions consist of a numerical coefficient c_M^k and a product of derivatives G of the one-link generating functional F (1.12b). The order of the derivatives are suppressed and so are the color indices. The link on which the variables in G sit is denoted by $(n + b_i, v_i)$, where v_i is one of the lattice directions $\mu_1,\ldots,\mu_l$ and b_i a finite linear combination

$$b = \sum_{j=1}^{l} d_j a^{\mu_j} \tag{2.2}$$

of unit lattice vectors in these directions.

Form (2.1) clearly holds for W_1 given in (1.13) and we shall now show it to hold for W_{k+1} if it does so for W_k.

Form (2.1) for W_k implies for the derivative of W_k on link (m, λ) that

$$\begin{aligned} \frac{\partial W_k}{\partial A(m,\lambda)} &= \sum_n \sum_M c_M^k \sum_{\mu_1,\ldots,\mu_l} \sum_j \delta_{v_j}^{\lambda} \delta(n + b_j - m) \prod_i G(n + b_i, v_i) \\ &= \sum_M c_M^k \sum_{\mu_1,\ldots,\mu_l} \sum_j \delta_{v_j}^{\lambda} \prod_i G(m - b_j + b_i, v_i). \end{aligned} \tag{2.3}$$

The same form holds for $\partial W_k / \partial \bar{A}(m, \kappa)$ and for derivatives on the other links of plaquette (m, κ, λ) if one substitutes for m the corresponding point m, $m + a^{\kappa}$, or $m + a^{\lambda}$. Higher order derivatives also have form (2.3). Substituting this form for the derivatives into recursion relation (1.10) then leads to the same form (2.1) for W_{k+1}. Thus in each order there is only one summation over the lattice points. This one summation can be suppressed altogether in the actual calculation.

The next point one has to consider is the proliferation of lattice direction

summations. It is clear that in four-dimensional Euclidean space one can keep the number of summation directions down to at most four by using exclusive sums $\sum'$, where the summation runs over different directions only. The way these arise in the calculation and rules to manipulate them are given in the next section. From these rules and the structure of recursion relation (1.10) it will be shown there that any term in a W_k for $k \neq 0$ has at least two such summation directions.

In view of manipulations with exclusive lattice direction sums, it turns out that the most efficient way to code the lattice vectors b in (2.1) is in terms of a variable basis $a^{\mu_1},..., a^{\mu_l}$ $(2 \leqslant l \leqslant 4)$ depending on the actual and exclusive directions $\mu_1,..., \mu_l$, as was already indicated in expression (2.2).

This completes the description of a term in W_k, which thus has the generic form

$$c_M^k \sum_{\mu_1,...,\mu_l}{}' \prod_i G(n + b_i, v_i), \qquad l \leqslant 4, \quad l \geqslant 2 \quad \text{if } k \neq 0. \tag{2.4}$$

Functional differentiation of such a term amounts to the execution of elementary algebraic operations such as substitution of lattice direction indices, shifts of lattice vectors, and appending pairs of color indices. This makes implementation on a computer feasible.

3. Detailed Structure of the Calculation

In this section we want to try to describe the calculation in such detail that, together with a manual on the operation of the programs that is presently being completed [6], the reader will be able to work with the results of the calculation or to redo or continue the calculation. For the first objective it is sufficient to understand the internal structural unit, a so-called W-bank, corresponding to a term (2.4). We explain the structure of the W-bank and the operations one must perform on them to get the higher order terms (differentiation, recombination) and close the section with a discussion on the spatial structure of the W_k terms, which provides us with bounds on some of the magnitudes occurring in the W-banks.

Let us first dwell briefly on the rôle the plaquettes play. In the recursion relation a new summation over plaquettes is introduced with each step in k. Thus W_k involves k such summations. As the result of differentiation and recombination they do not remain independent but are glued together. In the final result only the links on which the G's reside are relevant. Nevertheless the plaquettes help a great deal to obtain insight into the spatial extension of terms and they will be used for that purpose later. Furthermore, they can be used as a code both for the color indices in a derivative G and for distinguishing between differentiations with respect to A or $\bar{A}$. In order to make this possible, we keep the plaquettes as structural elements and number them (from 1 to k). Then the description of the derivatives G in terms of the one-link integrals F is done as follows: One of the elements of description is the link to which they belong. The other is a list of pairs (plaquette number, link number), where a pair (m_p, g) has the following meaning (cf. Fig. 1):

(1) Link g of plaquette m_p is the same as the link on which this G resides.

(2) For this g the type of derivative of F that has to be taken is defined in Fig. 1.

Note that the color indices are implicit, but from the following example it will be clear that they can be uniquely assigned from the information available. As an example, let plaquette m_p be (n, μ, ν) and the appended pair be $(m_p, 1)$. This means that the operation $\partial/\partial \bar{A}_a^d(n, \mu)$ has to be performed on F, where the upper color index a is the same as the lower color index on link 4 and the lower color index d is the same as the upper color index on link 2.

In the above we still denoted a lattice point by n or $n + b$, but as in any stage of the computation there is at most one lattice point summation involved, we may delete the reference to "n." Thus a point in storage is denoted by the lattice vector b. As we use a variable basis (see (2.2)), b is specified by 2 to 4 coefficients. These coefficients are bounded by the maximum number of lattice steps between any 2 points occurring in the term, a number that is shown at the end of the section to be $k + 1$ for a term in W_k.

Examples of W-banks are given in Fig. 2.

Let us now discuss the way exclusive summations arise and are treated. For $k \geqslant 1$ a complete term in W_k has exclusive lattice sums of the form

$$\sum_{\mu\nu}{}' f_{\mu\nu}, \qquad \sum_{\mu\nu\kappa}{}' f_{\mu\nu\kappa}, \qquad \sum_{\mu\nu\kappa\lambda}{}' f_{\mu\nu\kappa\lambda}. \tag{3.1}$$

Term W_1 involves only the first of these and as in each new order two new exclusive lattice summations are introduced in the recursion relation, the total number can never drop below two, and can by definition never exceed four. These terms have to be differentiated and later recombined by way of the recursion relation. Before the recombination is effected the lattice directions of the new plaquette in this order are treated as external indices, let us call them α, β. To perform the differentiation we found it useful to write out the exclusive lattice sums in terms of sums $\sum''$ that run only over indices that are mutually exclusive *and* different from α and β. This can be done in the three cases that may arise by

$$\sum_{\mu\nu}{}' f_{\mu\nu} = f_{\alpha\beta} + \sum_{\mu}{}'' (f_{\alpha\mu} + f_{\beta\mu} + f_{\mu\alpha} + f_{\mu\beta}) + \sum_{\mu\nu}{}'' f_{\mu\nu}, \tag{3.2a}$$

$$\begin{aligned}\sum_{\mu\nu\kappa}{}' f_{\mu\nu\kappa} = {} & \sum_{\mu}{}'' (f_{\alpha\beta\mu} + f_{\alpha\mu\beta} + f_{\mu\alpha\beta} + f_{\beta\alpha\mu} + f_{\beta\mu\alpha} + f_{\mu\beta\alpha}) \\ & + \sum_{\mu\nu}{}'' (f_{\alpha\mu\nu} + f_{\mu\alpha\nu} + f_{\mu\nu\alpha} + f_{\beta\mu\nu} + f_{\mu\beta\nu} + f_{\mu\nu\beta}),\end{aligned} \tag{3.2b}$$

$$\begin{aligned}\sum_{\mu\nu\kappa\lambda}{}' f_{\mu\nu\kappa\lambda} = {} & \sum_{\mu\nu}{}'' [f_{\alpha\beta\mu\nu} + f_{\alpha\mu\beta\nu} + f_{\alpha\mu\nu\beta} + f_{\mu\alpha\beta\nu} \\ & + f_{\mu\alpha\nu\beta} + f_{\mu\nu\alpha\beta} + (\alpha \rightleftarrows \beta)].\end{aligned} \tag{3.2c}$$

```
TERM      1  WITH K= 1   HAS
COEFFICIENT           1   AND  2  SUM.INDICES
PL  1 =(  0,  0),(1,2)
G(  1)=(  0,  0)*(1) ,( 1,1)
G(  2)=(  0,  0)*(2) ,( 1,4)
G(  3)=(  0,  1)*(1) ,( 1,3)
G(  4)=(  1,  0)*(2) ,( 1,2)


TERM      1  WITH K= 2   HAS
COEFFICIENT           1   AND  2  SUM.INDICES
PL  1 =(  0,  0),(1,2)   PL  2 =(  0,  0),(1,2)
G(  1)=(  0,  0)*(1) ,( 1,1),( 2,1)
G(  2)=(  0,  0)*(2) ,( 1,4),( 2,4)
G(  3)=(  0,  1)*(1) ,( 1,3),( 2,3)
G(  4)=(  1,  0)*(2) ,( 1,2),( 2,2)


TERM     50  WITH K= 2   HAS
COEFFICIENT           1   AND  3  SUM.INDICES
PL  1 =(  0,  0,  0),(1,2)   PL  2 =(  0,  0, -1),(1,3)
G(  1)=(  0,  0,  0)*(1) ,( 1,1),( 2,3)
G(  2)=(  0,  0,  0)*(2) ,( 1,4)
G(  3)=(  0,  0, -1)*(1) ,( 2,1)
G(  4)=(  0,  0, -1)*(3) ,( 2,4)
G(  5)=(  0,  1,  0)*(1) ,( 1,3)
G(  6)=(  1,  0,  0)*(2) ,( 1,2)
G(  7)=(  1,  0, -1)*(3) ,( 2,2)


TERM     32  WITH K= 4   HAS
COEFFICIENT           6   AND  3  SUM.INDICES
PL  1 =(  0,  0,  0),(1,2)   PL  2 =(  0,  0, -1),(2,3)
PL  3 =(  0,  0, -1),(3,2)   PL  4 =(  0,  1,  0),(1,2)
G(  1)=(  0,  0,  0)*(1) ,( 1,1)
G(  2)=(  0,  0,  0)*(2) ,( 1,4),( 2,3),( 3,2)
G(  3)=(  0,  0, -1)*(2) ,( 2,1),( 3,4)
G(  4)=(  0,  0, -1)*(3) ,( 2,4)
G(  5)=(  0,  0, -1)*(3) ,( 3,1)
G(  6)=(  0,  1,  0)*(1) ,( 1,3),( 4,1)
G(  7)=(  0,  1,  0)*(2) ,( 4,4)
G(  8)=(  0,  1, -1)*(3) ,( 2,2),( 3,3)
G(  9)=(  0,  2,  0)*(1) ,( 4,3)
G( 10)=(  1,  0,  0)*(2) ,( 1,2)
G( 11)=(  1,  1,  0)*(2) ,( 4,2)
```

FIG. 2. Sample terms from different W_k's. A plaquette is denoted by PL and specified by a lattice vector (the first doublet or triplet) and two lattice directions. A derivative of a one-link integral is denoted by G, and specified by: a lattice vector, a lattice direction, and a number of pairs (plaquette number, link number).

The first derivative then gets contributions from each term that contains the external index γ involved in the derivative ($\gamma = \alpha$ or β) explicitly. The number of contributions is equal to the number of G's occurring in that term, and in calculating the contribution associated with a specific G sitting on link (b, γ) one has to shift all other lattice points b' to $b' - b$ (or $b' - b + a^{\kappa}$, $\kappa = \alpha$ or β), cf. (2.3). Higher order derivatives only get contributions associated with a specific G if the link of that G coincides with the link of the differentiation.

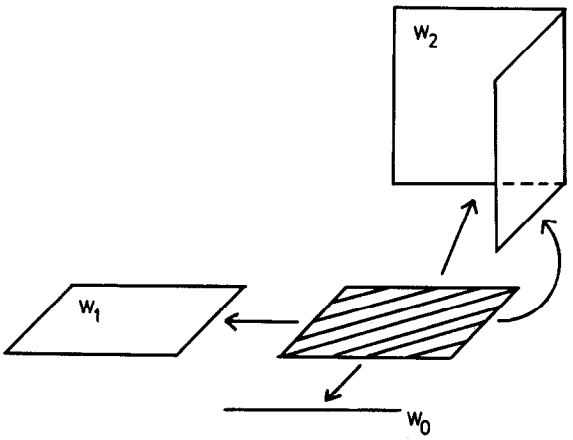

FIG. 3. Example of the way lower order terms (W_1 and W_2) are attached to a new plaquette so as to create a contribution to W_4.

To combine derivatives to a term of W_{k+1} by way of (1.10) one needs the rules to reduce a product of 2, 3, or 4 sums $\sum''$ to a sum of exclusive sums. These rules are easily deduced by means of formulae of the type

$$\left(\sum_{\mu\nu}'' f_{\mu\nu}\right)\left(\sum_{\kappa\lambda}'' g_{\kappa\lambda}\right) = \sum_{\mu\nu}'' f_{\mu\nu}(g_{\mu\nu} + g_{\nu\mu}). \tag{3.3}$$

In this formula the explicit dependence of f and g on α, β is suppressed. After all factors in the product forming a new term have been collected, α and β become summation indices again, turning $\sum_{\mu\nu}''$ into $\sum_{\alpha\beta\mu\nu}'$ and so on.

We conclude this section with a discussion on the spatial structure of terms of W_k. Differentiation of a $W_{k'}$ associated with a link of the new plaquette singles out one link of $W_{k'}$ and equates it to the link of the new plaquette as is illustrated in Fig. 3. Higher order derivatives couple still more links of $W_{k'}$ to the new plaquette (W_2 in Fig. 3). Thus the points of a term of W_k form a structure that is built from k plaquettes connected by common links.

The distance between two points in a term of W_k is the minimum number of lattice steps between them. Let the maximum distance in W_k be denoted by M_k. One can easily convince oneself from the above discussion that $M_k \leqslant M_{k'} + M_{k''}$, $k' + k'' \leqslant k - 1$. This implies, together with $M_1 = 2$, that $M_k \leqslant k + 1$. And as a linear chain of plaquettes satisfies this bound and gives rise to a possible term, $M_k = k + 1$. Thus a linear bound on the coefficients in the lattice vectors b is obtained.

TABLE I

Number of Terms in W_k

Order k	Number of terms
0	1
1	1
2	54
3	4561
4	583492

4. Results

Though the calculation of higher order terms is quite straightforward, both time and storage problems restrict pursuing the strong coupling series very far. Calculations have been performed on CDC machines of type 6400 and Cyber 173. The calculation of terms up to third order takes less than one CP-hour, but the fourth order takes a couple hundred CP-hours on the Cyber 173. This is a result of the explosive exponential increase in the number of terms in successive orders, as can be seen in Table I. The length of a term increases to 32 (60-bit) computer words in fourth order. The number of terms is understandable from the fact that order k involves k plaquettes and the resulting terms contain full spatial information. The amount of tape needed to store the results up to fourth order is somewhat more than a full (2400 foot) high density (6250 bpi) tape.

The programs involved in the calculations are adapted to the 60-bit word length of the CDC machines. A manual is presently being drawn up [6].

The results at this stage are independent of the gauge group and so other one-link integrals may be substituted; e.g., Z_N integrals have been used.

As was mentioned in the Introduction, our first goal is to substitute the known expressions for the derivatives of the one-link integrals for $SU(3)$ (and $U(3)$) in order to get the effective Lagrangian in terms of meson and baryon fields. This work is in progress.

Acknowledgments

The author is indebted to the NIKHEF-H for the use of computer facilities and to the members of the operating staff for their extensive help; and would like to thank Jan Smit for suggesting the problem, for discussions, and for partially testing the results.

REFERENCES

1. N. KAWAMOTO AND J. SMIT, *Nucl. Phys. B* **192** (1981), 100.
2. J. HOEK, N. KAWAMOTO, AND J. SMIT, *Nucl. Phys. B* **199** (1982), 495.
3. I. BARS AND F. GREEN, *Phys. Rev. D* **20** (1979), 3311; I. BARS, *J. Math. Phys.* **21** (1980), 2678; E. BRÉZIN AND D. J. GROSS, *Phys. Lett. B* **97** (1980), 120; R. C. BROWER AND M. NAUENBERG, *Nucl. Phys. B* **180** (FS2) (1981), 221; R. C. BROWER, P. ROSSI, AND C. I. TAN, *Phys. Rev. D* **23** (1981), 942; V. A. FATEEV AND E. ONOFRI, *Lett. Math. Phys.* **5** (1981), 367; K. E. ERIKSSON, N. SVARTHOLM, AND B. S. SKAGERSTAM, *J. Math. Phys.* **22** (1981), 2276. R. C. BROWER, P. ROSSI, AND C. I. TAN, *Nucl. Phys. B* **190** (FS3) (1981), 699.
4. J. HOEK, *Phys. Lett. B* **102** (1981), 129.
5. K. G. WILSON, *in* "New Phenomena in Subnuclear Physics, Erice Lectures 1975" (A. Zichichi, Ed.), Plenum, New York, 1977.
6. J. HOEK, "SCELFE Manual," Internal report, ITFA 82–4.